Universality of *S*-matrix correlations for deterministic plus random Hamiltonians

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We study *S*-matrix correlations for random matrix ensembles with a Hamiltonian $H=H_0+\varphi$, in which H_0 is a deterministic $N\times N$ matrix and φ belongs to a Gaussian random matrix ensemble. Using Efetov's supersymmetry formalism, we show that in the limit $N\to\infty$ correlation functions of *S*-matrix elements are universal on the scale of the local mean level spacing: the dependence of H_0 enters into these correlation functions only through the average *S* matrix and the average level density. This statement applies to each of the three symmetry classes (unitary, orthogonal, and symplectic).

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I. INTRODUCTION

The energy levels and/or the scattering matrices of a variety of physical systems with randomness (e.g., complex nuclei, disordered conductors, classically chaotic systems, etc.) exhibit universal behavior: the statistical properties of the observables can be separated into the universal parts and the nonuniversal parts specific to individual systems. There has been increasing evidence that the universal parts depend only on the fundamental symmetries of the underlying Hamiltonian and are well described by a random matrix ensemble with a Gaussian distribution (see Ref. [1] for a review). According to the fundamental symmetries, there are three classical random matrix ensembles: Systems with broken time-reversal symmetry are described by the unitary ensemble and time-reversal invariant systems by either the symplectic or the orthogonal ensemble depending on whether spin-orbit coupling is present or not [2]. In spite of their many successful applications, random matrix models lack a firm foundation. In particular, the Gaussian form of the probability distribution is used for mathematical convenience and is not motivated by physical principles. It is therefore necessary and important to investigate whether statistical properties are identical for more general forms of the probability distribution consistent with fundamental symmetries.

There has been some work along this direction. Hackenbroich and Weidenmüller [3] considered a non-Gaussian and unitary invariant probability distribution: $P(H)\propto\exp[-N\,\mathrm{tr}\,V(H)]$, where N is the dimension of the Hamiltonian matrix H and $V(H)$ is independent of N and arbitrary provided it confines the spectrum to some finite interval and generates a smooth mean level density, in the limit $N\to\infty$. For each of the three symmetry classes, using Efetov's supersymmetry formalism, they showed that both energy level correlation functions and correlation functions of *S*-matrix elements are independent of $P(H)$ and hence universal if the arguments of the correlators are scaled correctly.

For realistic situations, it is likely that the Hamiltonian is not completely random but contains some regular parts. If the total Hamiltonian $H=H_0+\varphi$ where H_0 is a deterministic part and φ is a random one, the probability distribution takes a unitary noninvariant form:

$$P(H)\propto\exp[-N\,\mathrm{tr}\,V(\varphi)]=\exp[-N\,\mathrm{tr}\,V(H-H_0)]. \tag{1.1}$$

For the unitary ensemble, Brézin *et al.* [4] discussed the universality of two-point energy level correlations for $V(\varphi)=\varphi^2/2+g\varphi^4$. General *n*-point energy level correlation functions were shown to be universal by Brézin and Hikami [5] for $V(\varphi)\propto\varphi^2$. (The other type of unitary noninvariant distribution $P(H)\propto\exp\{-N\,\mathrm{tr}[V(H)-HH_0]\}$ was also considered by Zinn-Justin [6].)

Recently, we [7] numerically found the same universality of the *S*-matrix correlations for the distribution function Eq. (1.1) with $V(\varphi)\propto\varphi^2$ for the orthogonal ensemble, i.e., with the average *S* matrix $\overline{S}$ taken as the parameter, the correlations are independent of H_0 while $\overline{S}$ depends on H_0. Our purpose in the present Brief Report is to analytically show this universality in all three symmetry classes. More precisely, we show that

$$\overline{\prod_{i=1}^{m}\prod_{j=1}^{n}\left[S_{a_ib_i}\left(E-\frac{\omega}{2}\right)\right]^{k_i}\left[S^{*}_{c_jd_j}\left(E+\frac{\omega}{2}\right)\right]^{l_j}}=f_\beta(\omega\rho(E),\overline{S}(E)), \tag{1.2}$$

where m, n, k_i, l_j are non-negative integers; the overbar denotes the ensemble average. The universal functions f_β depend on the symmetry classes ($\beta=1$, 2, and 4 for orthogonal, unitary, and symplectic classes) and are independent of H_0, except for the indices $\{a_i,b_i,k_i,c_j,d_j,l_j\}$, while the average local level density ρ and the average *S* matrix $\overline{S}$ depend on H_0.

II. THE MODEL

Following the approach of Ref. [8], we write the scattering matrix $S(E)$ as

$$S_{ab}(E)=\delta_{ab}-2i\pi\sum_{\mu,\nu}W^{\dagger}{}_{a\mu}[D(E)^{-1}]_{\mu\nu}W_{\nu b}, \tag{2.1a}$$

in which

$$D(E)=E+i0^{+}-H+i\pi WW^{\dagger}, \tag{2.1b}$$

where E is the energy, 0^+ is a positive infinitesimal, H represents the projection of the full Hamiltonian onto the interaction region, and W describes the coupling between the eigenstates of the interaction region and the scattering states in the free-propagation region. The indices a,b refer to the physical scattering channels, and μ,ν refer to the complete orthonormal states characterizing the interaction region.

We assume that the $N\times N$ matrix H can be written as

$$H=H_0+\varphi, \tag{2.2}$$

where H_0 is a given, nonrandom, Hermitian matrix, and φ is a member of the Gaussian ensemble. The symmetry property of H_0 is the same as that of φ. The independent elements of the matrix φ are uncorrelated random variables with a Gaussian probability distribution centered at zero. The second moments for the unitary ensemble are given by

$$\overline{\varphi_{\mu\nu}\varphi_{\mu'\nu'}}=\frac{\lambda^2}{N}\delta_{\mu\nu'}\delta_{\nu\mu'}. \tag{2.3}$$

(See Ref. [9] for the orthogonal and symplectic cases.) Here, λ is a strength parameter.

III. DERIVATION

For definiteness, we show the derivation for the unitary ensemble and $m,n\leqslant 2$ in Eq. (1.2). The generalization to the other symmetry classes and/or higher values of m and n is straightforward and commented upon in Sec. IV. The derivation is based on the use of Efetov's supersymmetry method [8,10]. We take the notation from Ref. [8] and use the [1,2] block notation for the matrix representation in which 1 and 2 refer to the retarded and advanced block, respectively.

Consider the following generating function:

$$Z(J)=\frac{\det[D_p(E_p)+2\pi WJ_p(F)W^\dagger]}{\det[D_p(E_p)-2\pi WJ_p(B)W^\dagger]}, \tag{3.1}$$

where $D_p(E_p)=\mathrm{diag}[D(E_1),D^\dagger(E_2)]$, $J_p(F)=\mathrm{diag}[J_1(F),J_2(F)]$, and $J_p(B)=\mathrm{diag}[J_1(B),J_2(B)]$. The scattering matrix can be generated from $Z(J)$ as follows:

$$S_p(E_p)_{ab}=\delta_{ab}-i\left.\frac{\partial Z(J)}{\partial J_p(B)_{ba}}\right|_{J=0}L=\delta_{ab}-i\left.\frac{\partial Z(J)}{\partial J_p(F)_{ba}}\right|_{J=0}L, \tag{3.2}$$

where $S_p(E_p)=\mathrm{diag}[S(E_1),S^\dagger(E_2)]$ and $L=\mathrm{diag}(1,-1)$. Using standard procedure [8], we can represent the average of $Z(J)$ as an integral over a 4×4 graded matrix field σ:

$$\overline{Z}(J)=\int d[\sigma]\exp\{\mathcal{L}(\sigma)\}, \tag{3.3a}$$

where

$$\mathcal{L}(\sigma)=-\frac{N}{2\lambda^2}\mathrm{trg}(\sigma^2)-\mathrm{trg}\ln\Big[\mathcal{D}(\sigma)-\frac{\omega^-}{2}L+i\pi WW^\dagger L -2\pi L_gWJ_p(g)W^\dagger\Big]. \tag{3.3b}$$

Here, trg denotes the graded trace, $\mathcal{D}(\sigma)=E-\sigma-H_0$, $\omega^-=\omega-i0^+$, $J_p(g)=\mathrm{diag}[J_1(B),J_1(F),J_2(B),J_2(F)]$, and $L_g=\mathrm{diag}(1,-1,1,-1)$. We have defined $E=(E_1+E_2)/2$ and $\omega=E_2-E_1$.

In the limit $N\to\infty$, this integral can be done with the use of the saddle-point approximation. We are interested in correlations involving energy differences ω of the order of the mean level spacing $\sim O(N^{-1})$. Hence, we expand $\mathcal{L}(\sigma)$ in powers of ω:

$$\begin{aligned}\mathcal{L}(\sigma)\approx&-\frac{N}{2\lambda^2}\mathrm{trg}(\sigma^2)-\mathrm{trg}\ln[\mathcal{D}(\sigma)]\\&-\mathrm{trg}\ln[1+i\pi\mathcal{D}(\sigma)^{-1}WW^\dagger L]\\&-\mathrm{trg}\ln\{1-2\pi[\mathcal{D}(\sigma)+i\pi WW^\dagger L]^{-1}L_gWJ_p(g)W^\dagger\}\\&+\frac{\omega^-}{2}\mathrm{trg}[\mathcal{D}(\sigma)^{-1}L].\end{aligned} \tag{3.4}$$

It should be noted that such an expansion is not possible for $WW^\dagger$ because $W^\dagger W\sim O(1)$. Of the five terms in expression (3.4) the last three terms are $O(1)$. The first two terms are $O(N)$ and determine the saddle point σ^{sp}. To derive the saddle-point equation we write H_0 and σ^{sp} in the forms $H_0=U^{-1}\mathrm{diag}(\epsilon_1,\ldots,\epsilon_N)U$ and $\sigma^{sp}=T^{-1}\sigma_D^{sp}T$, where σ_D^{sp} is diagonal and T has the form

$$T=\begin{pmatrix}(1+t_{12}t_{21})^{1/2} & it_{12}\\ -it_{21} & (1+t_{21}t_{12})^{1/2}\end{pmatrix}. \tag{3.5}$$

The saddle-point equation reads

$$\sigma_D^{sp}=\frac{\lambda^2}{N}\sum_{\mu=1}^{N}\frac{1}{E-\sigma_D^{sp}-\epsilon_\mu}. \tag{3.6}$$

For ordinary variables (rather than matrices), Eq. (3.6) has $N+1$ solutions, $N-1$ of which are real, and the remaining two may have nonzero imaginary parts according to the values of E. Taking the two complex solutions $(r\pm i\Delta)$ [5], we obtain $\sigma_D^{sp}=r-i\Delta L$. The explicit expressions of r and Δ are not available because Eq. (3.6) becomes in general an $(N+1)$th polynomial. Several authors have discussed the properties of Eq. (3.6) (see, e.g., [4,11]). Hereafter we consider the case where $\Delta\sim O(1)$. From the relation between Δ and the average level density ρ [Eq. (3.8a)], this means that E lies far away from the edge of the spectrum. Substituting σ^{sp} for σ in Eq. (3.4), we find

$$\mathcal{L}(\sigma^{sp})\approx -\mathrm{trg}\ln[1+i\pi\mathcal{D}(\sigma^{sp})^{-1}WW^{\dagger}L] + \frac{\omega^{-}}{2}\,\mathrm{trg}[\mathcal{D}(\sigma^{sp})^{-1}L] - \mathrm{trg}\ln\{1-2\pi W^{\dagger}[\mathcal{D}(\sigma^{sp}) + i\pi WW^{\dagger}L]^{-1}WL_g J_p(g)\}. \tag{3.7}$$

The one-point functions $\rho(E)$ and $\overline{S}(E)$ are evaluated at the saddle point. We thus have

$$\rho(E)=\frac{N\Delta}{\pi\lambda^2} \tag{3.8a}$$

and

$$\overline{S_p}(E)=1-2i\pi W^{\dagger}[\mathcal{D}(\sigma_D^{sp})+i\pi WW^{\dagger}L]^{-1}WL. \tag{3.8b}$$

Using these one-point functions $\rho(E)$ and $\overline{S}(E)$ we can write each term of Eq. (3.7) as follows:

$$\mathrm{trg}\ln[1+i\pi\mathcal{D}(\sigma^{sp})^{-1}WW^{\dagger}L]=\mathrm{trg}\ln\{1-[\overline{S_p}(E)-1]LM\}, \tag{3.9a}$$

$$\frac{\omega^{-}}{2}\mathrm{trg}[\mathcal{D}(\sigma^{sp})^{-1}L]=-2i\pi\omega^{-}\rho(E)\mathrm{trg}(t_{12}t_{21}), \tag{3.9b}$$

and

$$2\pi W^{\dagger}[\mathcal{D}(\sigma^{sp})+i\pi WW^{\dagger}L]^{-1}W = i\{T^{-1}L[\overline{S_p}(E)-1]^{-1}T-T^{-1}MT\}^{-1}. \tag{3.9c}$$

Here,

$$M=\begin{pmatrix} t_{12}t_{21} & -it_{12}(1+t_{21}t_{12})^{1/2} \\ -it_{21}(1+t_{12}t_{21})^{1/2} & -t_{21}t_{12}\end{pmatrix} \tag{3.10}$$

and we used the property $TLT^{-1}=L+2M$. More explicitly, Eqs. (3.9a) and (3.9c) can be expressed with the use of t_{12} and t_{21} as follows:

$$\mathrm{trg}\ln[1+i\pi\mathcal{D}(\sigma^{sp})^{-1}WW^{\dagger}L]=\mathrm{trg}\ln(1+\mathcal{T}_{12}t_{12}t_{21}) =\mathrm{trg}\ln(1+\mathcal{T}_{21}t_{12}t_{21}) \tag{3.11a}$$

and

$$2\pi W^{\dagger}[\mathcal{D}(\sigma^{sp})+i\pi WW^{\dagger}L]^{-1}W=i\begin{pmatrix}\overline{S}(E)(1+\mathcal{T}_{21}t_{12}t_{21})^{-1}-1 & -it_{12}(1+t_{21}t_{12})^{1/2}(\mathcal{T}_{12}^{-1}+t_{21}t_{12})^{-1} \\ -it_{21}(1+t_{12}t_{21})^{1/2}(\mathcal{T}_{21}^{-1}+t_{12}t_{21})^{-1} & 1-\overline{S}^{\dagger}(E)(1+\mathcal{T}_{12}t_{21}t_{12})^{-1}\end{pmatrix}, \tag{3.11b}$$

where $\mathcal{T}_{12}=1-\overline{S}(E)\overline{S}^{\dagger}(E)$ and $\mathcal{T}_{21}=1-\overline{S}^{\dagger}(E)\overline{S}(E)$ [12]. (The derivation is given in the Appendix.) Thus we find that all the dependence of Eq. (3.7) on W and H_0 is completely absorbed in $\rho(E)$ and $\overline{S}(E)$. Equations (3.9a), (3.9b), and (3.9c) show universality. The explicit forms of the correlation functions f_β are identical with those for the Gaussian ensemble and can be found in appropriate references (see, e.g., [1,13]).

IV. SUMMARY

For the sake of simplicity, we presented the derivation for only the unitary ensemble. In either the orthogonal or the symplectic ensemble, the internal structures of t_{12} and t_{21} differ from the unitary case. However, our derivaton is completely independent of such structures and applies equally to the orthogonal and symplectic ensembles. Taking the generating function

$$Z(J)=\prod_{q=1}^{\max\{m,n\}}\frac{\det[D_p(E_p)+WJ_p^q(F)W^{\dagger}]}{\det[D_p(E_p)-WJ_p^q(B)W^{\dagger}]}, \tag{4.1}$$

we can show Eq. (1.2) for $m>2$ or $n>2$ along exactly parallel lines.

In summary, we have shown that *the local universality in the bulk scaling limit* still holds for the *S*-matrix correlation functions even though unitary invariance is broken by the addition of a deterministic matrix to the ensemble. The starting random matrix model contains parameters W and H_0 that are specific to individual systems. After ensemble averaging, these original parameters are completely absorbed into parameters $\overline{S}(E)$ and $\rho(E)$. Thus *S*-matrix correlaton functions of the type Eq. (1.2) have universal forms that are independent of H_0 but for $\overline{S}(E)$ and $\rho(E)$ and are determined only by the symmetry of the ensemble. This holds for all three symmetry classes (orthogonal, unitary, and symplectic). The derivation can be similarly applied to the spectral correlation functions. Thus we have extended the previous results by Brézin and Hikami [5] to the orthogonal and symplectic ensembles although only two-point functions are considered.

The present results were derived under the restrictions that the correlation functions contain only two values of energy, E_1 and E_2, and that $V(\varphi)$ has a Gaussian form. It is a natural conjecture that the universality of the S-matrix correlation functions holds even if these two restrictions are removed. The increase of the number of energy arguments makes the structure of σ_D^{sp} more complicated. With this point taken properly into account, a similar derivation is probable. The extension to the general form of $V(\varphi)$ seems less trivial because we are no longer able to use a Hubbard-Stratonovich transformation in order to introduce a graded matrix σ. The simlar procedure used in Ref. [3] may be incorporated into the present derivation.

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APPENDIX: DERIVATION OF EQS. (3.11a) AND (3.11b)

To derive Eqs. (3.11a) and (3.11b), we use the fact that for any analytic function F we have $t_{12}F(t_{21}t_{12}) = F(t_{12}t_{21})t_{12}$. Using the identity

$$\mathrm{trg}\,\ln\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{trg}\,\ln(a - bd^{-1}c) + \mathrm{trg}\,\ln(d), \tag{A1}$$

we obtain Eq. (3.11a). With the abbreviation $Y \equiv \{T^{-1}L[\overline{S_p}(E)-1]^{-1}T - T^{-1}MT\}^{-1}$, using the property $T^{-1} = LTL$, we get the equation

$$Y = \begin{pmatrix} [\overline{S}(E)-1]^{-1} + t_{12}t_{21}A & it_{12}(1+t_{21}t_{12})^{1/2}A \\ it_{21}(1+t_{12}t_{21})^{1/2}A & -[\overline{S^\dagger}(E)-1]^{-1} - t_{21}t_{12}A \end{pmatrix}^{-1}, \tag{A2}$$

where $A = [\overline{S}(E)-1]^{-1} + [\overline{S^\dagger}(E)-1]^{-1} + 1$. With the use of the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} (a-bd^{-1}c)^{-1} & -a^{-1}b(d-ca^{-1}b)^{-1} \\ -d^{-1}c(a-bd^{-1}c)^{-1} & (d-ca^{-1}b)^{-1} \end{pmatrix}, \tag{A3}$$

the [1,1] block of Y can be written as follows:

$$Y_{11} = A^{-1}\{[\overline{S^\dagger}(E)-1]^{-1} + t_{12}t_{21}A\}\{[\overline{S}(E)-1]^{-1}A^{-1}[\overline{S^\dagger}(E)-1]^{-1} - t_{12}t_{21}\}^{-1}. \tag{A4}$$

Using the property

$$A = [\overline{S^\dagger}(E)-1]^{-1}[\overline{S^\dagger}(E)\overline{S}(E)-1][\overline{S}(E)-1]^{-1} = [\overline{S}(E)-1]^{-1}[\overline{S}(E)\overline{S^\dagger}(E)-1][\overline{S^\dagger}(E)-1]^{-1}, \tag{A5}$$

we obtain the [1,1] block of Eq. (3.11b). Similarly, the [2,1] block of Y can be written as follows:

$$Y_{21} = it_{21}(1+t_{12}t_{21})^{1/2}\{[\overline{S^\dagger}(E)-1]^{-1} + t_{12}t_{21}A\}^{-1}AY_{11}. \tag{A6}$$

Substituting Eq. (A4) for Y_{11} in Eq. (A6), we obtain the [2,1] block of Eq. (3.11b). The other blocks are obtained along exactly parallel lines.

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